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1986 J. Phys. A: Math. Gen. 19 L473

(http://iopscience.iop.org/0305-4470/19/9/003)

J. Phys. A: Math. Gen. 19 (1986) L473–L478. Printed in Great Britain

LETTER TO THE EDITOR

Ladder operator relations for hypergeometric functions

U Laha, C Bhattacharyya and B Talukdar

Department of Physics, Visva-Bharati University, Santiniketan-731 235, West Bengal, India

Received 1 April 1986

Abstract. Two first-order differential operators are introduced to generate recursion formulae for hypergeometric functions. These operators, on the one hand, factorise the associated second-order operator and, on the other hand, reproduce the equations resulting from the action of the supersymmetry generators on the positive-energy solution of the Coulomb field. Our results for the confluent hypergeometric functions also provide a natural basis for constructing ladder operator recursion relations for the Coulomb Green function with the outgoing wave boundary condition.

In this letter we derive a simple factorisation method for differential equations satisfied by hypergeometric functions. We achieve this by constructing a pair of first-order differential operators, $O^{(\mp)}$. Our construction procedure is, however, different from that of Infeld and Hull [1]. Interestingly, the operators $O^{(\mp)}$ will be seen to generate ladder operator relations which have their counterparts in supersymmetric quantum mechanics [2]. Furthermore, the results for the confluent hypergeometric functions will be useful in constructing expressions for raising and lowering operators for the Coulomb Green function with the outgoing wave boundary condition.

The confluent hypergeometric function satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}y}{\mathrm{d}x}-ay=0. \tag{1}$$

The regular and irregular solutions of equation (1) are given by [3]

$$y_1=\Phi(a;\ c;\ x)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!} \tag{2}$$

and

$$y_2=\Psi(a;\ c;\ x)=\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\Phi(a;\ c;\ x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a+1-c;\ 2-c;\ x). \tag{3}$$

To derive a factorisation method for the second-order differential operator in equation (1) we proceed as follows.

Lemma 1. The differential operators

$$O^{(\mp)}=-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x}\ln\xi^{(\mp)}(x) \tag{4}$$

represent the raising and lowering operators for $\Phi(\cdot)$ with

$$\xi^{(-)}(x)=\lim_{x\to 0}\Phi(a;\ c;\ x) \tag{5}$$

and

$$\xi^{(+)}(x)=\lim_{x\to 0}\Psi(a;\ c;\ x). \tag{6}$$

Proof. From equations (1) and (3) one can immediately see that

$$\xi^{(-)}(x) \underset{x\to 0}{\sim} \exp(ax/c) \tag{7}$$

and

$$\xi^{(+)}(x) \underset{x\to 0}{\sim} x^{1-c}\exp\left(\frac{(1+a-c)}{(2-c)}x\right). \tag{8}$$

We now use equations (4), (7) and (8) to construct explicit expressions for $O^{(\mp)}$. From the results for $O^{(\mp)}\Phi(\cdot)$ we can write

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{a}{c}\right)\Phi(a;\ c;\ x)=-\frac{a(c-a)}{c^2(c+1)}x\Phi(a+1;\ c+2;\ x) \tag{9}$$

and

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1-c}{x}+\frac{1+a-c}{2-c}\right)\Phi(a;\ c;\ x)=\frac{1-c}{x}\Phi(a-1;\ c-2;\ x). \tag{10}$$

In deriving equations (9) and (10) we have made use of equation (2) together with the well known differential property of $\Phi(a;\ c;\ x)$. Equations (9) and (10) prove the lemma.

Lemma 2. The operators in equation (4), or equivalently those in (9) and (10), are also the raising and lowering operators for $\Psi(a;\ c;\ x)$.

Proof. Using the differential property of $\Psi(a;\ c;\ x)$ we have

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{a}{c}\right)\Psi(a;\ c;\ x)=a\Psi(a+1;\ c+1;\ x)+\frac{a}{c}\Psi(a;\ c;\ x). \tag{11}$$

The right-hand side of equation (11) can be reduced to a single $\Psi(\cdot)$ by employing the recursion relations [3]

$$\Psi(a;\ c+1;\ x)=\Psi(a;\ c;\ x)-\frac{\mathrm{d}}{\mathrm{d}x}\Psi(a;\ c;\ x) \tag{12}$$

and

$$(c-a)\Psi(a;\ c;\ x)+\Psi(a-1;\ c;\ x)-x\Psi(a;\ c+1;\ x)=0. \tag{13}$$

We obtain

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{a}{c}\right)\Psi(a;\ c;\ x)=\frac{a}{c}x\Psi(a+1;\ c+2;\ x). \tag{14}$$

Again,

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1-c}{x}+\frac{1+a-c}{2-c}\right)\Psi(a;\ c;\ x)$$
$$=a\Psi(a+1;\ c+1;\ x)+\left(\frac{1-c}{x}+\frac{1+a-c}{2-c}\right)\Psi(a;\ c;\ x). \tag{15}$$

Equation (15) can be combined with the integral representation [3]

$$\Psi(a;\ c;\ x)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-xt}\,t^{a-1}(1+t)^{c-a-1}\,\mathrm{d}t \tag{16}$$

to write

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1-c}{x}+\frac{1+a-c}{2-c}\right)\Psi(a;\ c;\ x)=\frac{c-a-1}{(c-2)x}\Psi(a-1;\ c-2;\ x). \tag{17}$$

Equations (14) and (17) prove the lemma.

It is straightforward to verify that the first-order differential-difference equations (9) and (10) are equivalent to the second-order equation in (1). For $a\to a+1$ and $c\to c+2$ equation (10) reads

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+(c+1)+\frac{a-c}{c}x\right)\Phi(a+1;\ c+2;\ x)=(c+1)\Phi(a;\ c;\ x). \tag{18}$$

Operating $[x(\mathrm{d}/\mathrm{d}x)+(c+1)+(a-c/c)x]$ on equation (9) we obtain

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+(c+1)+\frac{a-c}{c}x\right)\left(-\frac{\mathrm{d}\Phi(a;\ c;\ x)}{\mathrm{d}x}+\frac{a}{c}\Phi(a;\ c;\ x)\right)$$
$$=-\frac{a(c-a)}{c^2(c+1)}x\left(x\frac{\mathrm{d}}{\mathrm{d}x}+(c+2)+\frac{a-c}{c}x\right)\Phi(a+1;\ c+2;\ x). \tag{19}$$

Equations (9), (18) and (19) can now be combined to obtain equation (1).

The recipe in equation (4) can also be used to generate ladder operator relations for the Gaussian hypergeometric function and the associated factorisation is obvious. The Gaussian function satisfies the differential equation

$$\left(x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+[c-(a+b+1)x]\frac{\mathrm{d}}{\mathrm{d}x}-ab\right)y=0. \tag{20}$$

From equation (20) behaviour of the regular function $y_1=F(a,b;\ c;\ x)$ near the origin is obtained as

$$\xi^{(-)}(x)\underset{x\to 0}{\sim}\exp(abx/c) \tag{21}$$

while that for the irregular function $y_2=x^{1-c}F(1+a-c,1+b-c;\ 2-c;\ x)$ is of the form

$$\xi^{(+)}(x)\underset{x\to 0}{\sim}x^{1-c}\exp\left(\frac{(1+a-c)(1+b-c)}{(2-c)}x\right). \tag{22}$$

We now construct the differential operators $O^{(\mp)}$ from equations (4), (21) and (22), and operate them appropriately on $F(a, b; c; x)$. After some manipulation we arrive at

$$\left[-\left(1+\frac{1}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{ab}{cx}\right]F(a, b; c; x)$$
$$=-\frac{ab}{c(c+1)}\left((a+b+c+2)-\frac{ab}{c}\right)F(a+1, b+1; c+2; x) \qquad (23)$$

and

$$\left(-x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{[(a+b)(1-c)+ab-1]}{2-c}x+(1-c)\right)F(a, b; c; x)$$
$$=(1-c)F(a-1, b-1; c-2; x). \qquad (24)$$

Thus the operators in equations (23) and (24) turn $F(a, b; c; x)$ into $F(a+1, b+1; c+2; x)$ and $F(a-1, b-1; c-2; x)$ respectively.

The Gaussian hypergeometric equation has three regular singularities at 0, 1 and ∞ while the confluent equation has two singularities, one of which is regular and the other irregular. The regular singularity occurs at zero and the irregular at infinity. The confluent hypergeometric equation can be derived from the Gaussian equation by transforming the independent variable by $bx = z$ and finally letting $b \to \infty$. This transformation has the effect of merging (confluence) the two regular singularities at $x = 0$ and 1 into one regular singularity at $z = 0$ and converting the singularity at infinity from a regular to an irregular one. In particular, we have [4]

$$\lim_{b\to\infty}[F(a, b; c; z/b)] = \Phi(a; c; z). \qquad (25)$$

In the following we derive equations (9) and (10) from (23) and (24) by invoking the method of confluence of singularities. The substitution $bx = z$ converts equations (23) and (24) into the forms

$$\left[-\left(\frac{1}{b}+\frac{1}{z}\right)\frac{\mathrm{d}}{\mathrm{d}z}+\frac{a}{cz}\right]F(a, b; c; z/b)$$
$$=-\frac{a}{c(c+1)}\left(\frac{a+c+2}{b}-\frac{a}{c}+1\right)F(a+1, b+1; c+2; z/b) \qquad (26)$$

and

$$\left\{z\left(\frac{z}{b}-1\right)\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{(2-c)}\left[\left(\frac{a}{b}+1\right)(1-c)-\frac{1}{b}+a\right]z+(1-c)\right\}F(a, b; c; z/b)$$
$$=(1-c)F(a-1, b-1; c-2; z/b). \qquad (27)$$

In the limit $b \to \infty$ equations (26) and (27) in conjunction with (25) yield the results in equations (9) and (10).

Most of the special functions in mathematical physics are connected by simple relations either with the confluent or with the Gaussian hypergeometric function. Thus equations (9), (10), (23) and (24) will also generate ladder operator recursion relations for the associated special functions. Alternatively, one can also proceed by using our ansatz in equation (4). With regard to the second viewpoint we cite the example of the incomplete gamma function. For the incomplete gamma function $\gamma(a, x)$ satisfying the differential equation [3]

$$x\frac{d^2y}{dx^2}+(1+a+x)\frac{dy}{dx}+ay=0 \tag{28}$$

we have found that

$$\left(-\frac{d}{dx}+\frac{a}{x}\right)\gamma(a,x)=\frac{1}{x}\gamma(a+1,x). \tag{29}$$

Also, by considering the behaviour of

$$\Gamma(a,x)=e^{-x}\Psi(1-a;1-a;x) \tag{30}$$

we arrive at a lowering relation

$$\left(-\frac{d}{dx}-1\right)\gamma(a,x)=(1-a)\gamma(a-1,x). \tag{31}$$

Some physical applications of the results given above are now in order. In supersymmetric quantum mechanics (SQM) [2] one often deals with hierarchy problems. For example, within the framework of SQM we can generate a Hamiltonian hierarchy, the adjacent members of which are 'supersymmetric partners' in that they share the same eigenvalue spectrum except for the missing ground state [5]. In the case of the non-relativistic Kepler problem the Hamiltonian hierarchy corresponds to the addition of an appropriate centrifugal potential and the so-called accidental degeneracy is recovered as a natural consequence. The supersymmetry generators (generators for the 'supersymmetric partner' Hamiltonian and associated quantities) for the Coulomb field are given by [6]

$$Q_l^{(\mp)}=\mp\frac{d}{dr}+\frac{d}{dr}\ln\Psi_{l+1,l}^{(0)}(r) \tag{32}$$

with

$$\Psi_{l+1,l}^{(0)}(r)\sim r^{l+1}\exp\left(-\frac{e^2}{l+1}r\right) \tag{33}$$

the ground-state wavefunction obtained by using $n=l+1$. From equations (32) and (33) we have

$$Q_l^{(\mp)}=\mp\frac{d}{dr}+\frac{l+1}{r}-\frac{e^2}{l+1}. \tag{34}$$

It has been shown that the relations in equation (34) are valid for positive energies as well [7]. Introducing $E=+k^2/2$ and $\eta=-e^2/k$ we write the supersymmetry operators $Q_l^{(\mp)}$ for the positive-energy states as

$$\tilde{Q}_l^{(\mp)}=\mp\frac{d}{dr}+\frac{l+1}{r}+\frac{\eta k}{l+1}. \tag{35}$$

The action of the operators $\tilde{Q}_l^{(-)}$ and $\tilde{Q}_l^{(+)}$ upon the positive-energy Coulomb wavefunction [8]

$$f_{\eta,l}(k,r)=(2k)^l\exp(-\pi\eta/2)\frac{\Gamma(l+1+i\eta)}{\Gamma(2l+2)}r^{l+1}$$
$$\times\exp(-ikr)\Phi(l+1-i\eta;2l+2;2ikr) \tag{36}$$

yields

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k\eta}{l+1}+\mathrm{i}k\right)\Phi(l+1-\mathrm{i}\eta;\,2l+2;\,2\mathrm{i}kr)$$

$$=k^2\frac{[(l+1)^2+\eta^2]}{(l+1)^2(2l+3)}\,r\Phi(l+2-\mathrm{i}\eta;\,2l+4;\,2\mathrm{i}kr) \tag{37}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k\eta}{l}+\frac{2l+1}{r}-\mathrm{i}k\right)\Phi(l+1-\mathrm{i}\eta;\,2l+2;\,2\mathrm{i}kr)=\frac{2l+1}{r}\,\Phi(l-\mathrm{i}\eta;\,2l;\,2\mathrm{i}kr). \tag{38}$$

It is of interest to note that equations (37) and (38) are the special cases of (9) and (10) when $a=l+1-\mathrm{i}\eta$, $c=2l+2$ and $x=2\mathrm{i}kr$.

The radial Coulomb Green function $G_{\mathrm{cl}}^{(+)}(k;\,r;\,r')$ with outgoing wave boundary condition [8] is expressed in terms of regular and irregular confluent hypergeometric functions. Thus, given equations (9), (10), (14) and (17) one can obtain ladder operator recursion relations for $G_{\mathrm{cl}}^{(+)}(k;\,r,\,r')$ similar to those for the reduced Coulomb Green function [9].

This work forms part of a research project 'Some studies in few-body problems' supported by the Department of Atomic Energy, Government of India. The authors would like to thank Dr Ranabir Dutt for his kind interest in this work.

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